Announcements

1) #3 c), on HW #2
you may assume $\sqrt{6}$ is
irrational

2) Math Career Talks
Today after class
in CB 1030
2:30

Theorem $|\mathbb{R}| = |(0,1)|$

proof. In 2 steps:

1) $|(0,1)| = |(1,\infty)|$

Define a bijection $\varphi : (0,1) \to (1,\infty)$

by
$$\varphi(x) = \frac{1}{x}.$$

φ <u>is injective</u> Suppose

$\varphi(x) = \varphi(y)$. Then $\frac{1}{x} = \frac{1}{y}$,

so by cross-multiplication,

$x = y$

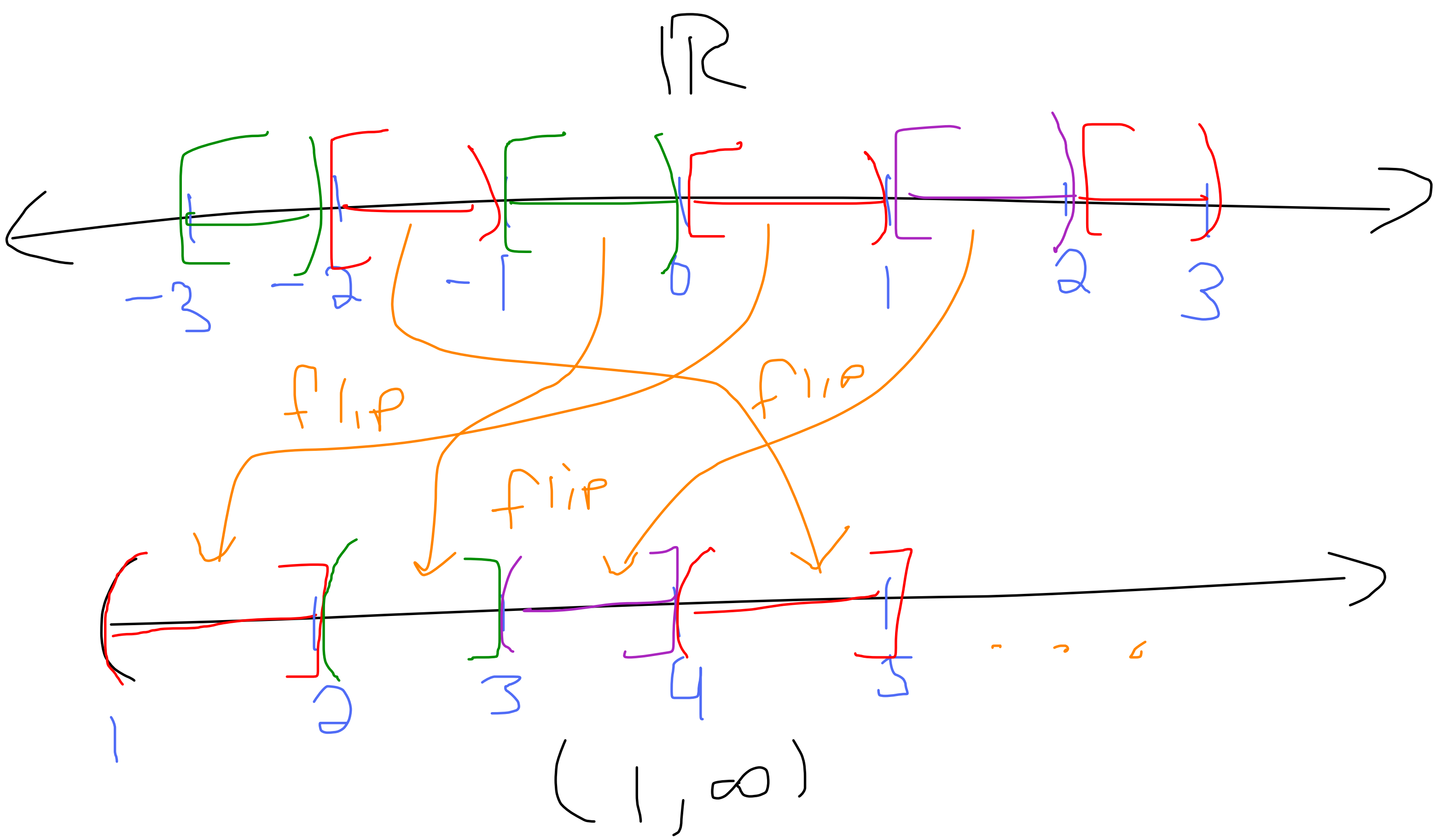
φ <u>is surjective</u> Let $y \in (1, \infty)$

Then with $x = \frac{1}{y}$, $0 < x < 1$,

and $\varphi(x) = \frac{1}{x} = \frac{1}{\left(\frac{1}{y}\right)} = y$.

Step 2: $|(1,\infty)| = |\mathbb{R}|$.

Define a bijection ψ from $\mathbb{R}$ to $(1,\infty)$ first with a picture!



Define, for $x \in \mathbb{R}$,

$\lfloor x \rfloor$ = the largest integer smaller than or equal to x.

e.g. $\lfloor \frac{1}{2} \rfloor = 0$ $\quad \lfloor -\frac{1}{2} \rfloor = -1$

$\lfloor \pi \rfloor = 3$

$\lfloor 4 \rfloor = 4$

$(\lfloor n \rfloor = n \quad \forall\, n \in \mathbb{Z})$

Define $\psi(x)$ as follows.

$$\psi(x) = \begin{cases} 3\lfloor x \rfloor + 2 - x, & x \geq 0 \\ 3\left|\lfloor x \rfloor\right| + 1 + x, & x < 0 \end{cases}$$

note: if $x \geq 0$, $\exists\, n \in \mathbb{N} \cup \{0\}$

$$n \leq x < n+1$$

If so, then for all such x,

$$\lfloor x \rfloor = n.$$

Then

$\psi(x) = 3n+2-x.$

now $n \leq x < n+1$, so

$$3n+2-(n+1) < \psi(x) \leq 3n+2-n$$

$$\parallel$$

$$2n+1 < \psi(x) \leq 2n+2$$

HW: Check that ψ is a bijection! $\square$

Corollary. $\mathbb{R}$ is not countable

proof. $(0,1)$ is not countable and $\mathbb{R}$ is in bijection with $(0,1)$. $\square$

We have answered one of our questions from last time. What about the other?

Is there a "hierarchy" of infinite sets?

So if $|X|$ is infinite, can we find Y with

$$|Y| \text{ ``}>\text{'' } |X| \, ?$$

Example 1: Let X be a finite set. How many subsets must X have?

Suppose $|X| = n < \infty$. Then X has 2^n distinct subsets

proof: By induction

Suppose $|X| = 1$. Then X has $2 = 2^1$ subsets, X and $\emptyset$.

Suppose we know the result for $|X| < n$. Suppose $|X| = n$.

Choose $x_0 \in X$ and let

$Y = X - \{x_0\}$. By induction,

$|Y| = n-1$, so Y has 2^{n-1}

subsets. Since any subset

$Z \subseteq X$ has the property that

either $x_0 \in Z$ or $x_0 \in Z^c$

but not both,

we obtain a bijection from the subsets of X containing x_0 to the subsets of X without x_0 by

$$Z \longmapsto Z^c.$$

Therefore, the number of subsets of X = $2 \cdot$ (the number of subsets of Y)

$$= 2 \cdot 2^{n-1} = 2^n \quad \square.$$

Definition: ($\mathcal{P}(X)$, 2^X notation)

If X is any set, define the **power set** of X to be

$$\mathcal{P}(X) = \{\text{subsets of } X\}$$

Alternate Notation: $2^X = \mathcal{P}(X)$

Starting Ch 2 next class

Read intro